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Abstract: Holder continuity and uniqueness of the solutions of general multivalued vector quasiequilibrium problems in metric spaces are established. The results are shown to be extensions of recent ones for equilibrium problems with some improvements. Applications in quasivariational inequalities, vector quasioptimization and traffic network problems are provided as examples among others in various optimization-related problems.

Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions

Lam Quoc Anh · Phan Quoc Khanh

Abstract Hölder continuity and uniqueness of the solutions of general multivalued vector quasiequilibrium problems in metric spaces are established. The results are shown to be extensions of recent ones for equilibrium problems with some improvements. Applications in quasivariational inequalities, vector quasi-optimization and traffic network problems are provided as examples for others in various optimization - related problems.

Keywords Multivalued vector quasiequilibrium problems · Hölder continuity · Solution uniqueness · Quasivariational inequalities · Traffic network problems · Quasioptimization

1 Introduction

The equilibrium problem, introduced by Blum and Oettli (1994) as a direct gen-

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eralization of variational inequalities and optimization problems, has been proved to include many optimization - related problems. However, it does not contain quasivariational inequalities. The origin of the latter is the paper of Bensoussan et al. (1973) considering random impulse control problems and showing the need to deal with constraint sets depending on the state variables. A natural extension of the equilibrium problem to include quasivariational inequalities is the quasiequilibrium problem, which contains also various quasioptimization - related problems. Up to now there have been a great deal of works devoted to all aspects of quasiequilibrium problems like the solution existence, the sensitivity analysis and stability, solving methods, the solution uniqueness, etc. For the sensitivity anallysis ans stability we observe Bianchi and Pini (2003), Anh and Khanh (2004, (JOTA) 2007 in press, submitted), which are devoted to semicontinuity of solution sets, and Ait Mansour and Riahi (2005), Anh and Khanh (2006, (JOGO) online 2006) which investigate the Hölder continuity of the unique solution of equilibrium problems.

The aim of the present paper is to extend Ait Mansour and Riahi (2005) and Anh and Khanh (2006 and (JOGO) online 06) to the case of general quasi-equilibrium problems. When applying to quasivariational inequalities in reflexive Banach spaces our result sharpens that of Adly et al. (in press). Since the solution existence has been intensively studied, (see e.g., resent papers Giannessi 2000; Goh and Yang 1999; Hai and Khanh 2006, 2007, (JOTA) in press, Khaliq 2005 and references therein), we focalize our consideration on sensitivity analysis assuming always that the solutions exist in a neighborhood of the reference point.

The outline of the remainder of the paper is as follows. The rest of this

section is devoted to explaining notions needed in the sequel. The main result is established in Section 2 followed by several direct sequences. In Section 3 we discuss applications of the main result.

Our notations are almost standard. We use $\|.\|$ and $d(.,.)$ for the norm and metric in any normed space and metric space, respectively, (the context makes it clear what space is encountered). $d(x,A)$ is the distance from x to subset A in X. For a normed space X, X^* is the topological dual and $\langle .,. \rangle$ is the canonical pair. R_+ is the set of nonnegative real numbers. $B(x,r)$ denotes the closed ball of radius $r \geq 0$ and centered at x in a metric space X. $\mathrm{int}C$ stands for the interior of a subset C.

Throughout the paper if not stated otherwise, let X, Z, Λ, M and N be metric spaces, Y be a metric linear space, $A \subseteq X$ be a nonempty subset and $C \subseteq Y$ have $\mathrm{int}C \neq \emptyset$. Let $K : A \times \Lambda \to 2^X$ be a multifunction with nonempty values, $a : A \times N \to 2^Z$ and $F : X \times X \times Z \times M \to 2^Y$ be multifunctions. For subsets A and B under consideration we adopt the notations

$$\mathrm{r}_1(A, B) \quad \text{means} \quad A \cap B \neq \emptyset;$$

$$\mathrm{r}_2(A, B) \quad \text{means} \quad A \subseteq B;$$

$$\varphi_1(A) = \big((-A) \setminus l(A)\big)^c;$$

$$\varphi_2(A) = (-\mathrm{int}A)^c,$$

where $l(A) = A \cap (-A)$ and $(.)^c$ is the complement of $(.)$. For each $\mathrm{r} \in \{\mathrm{r}_1, \mathrm{r}_2\}$, $\varphi \in \{\varphi_1, \varphi_2\}$, $\lambda \in \Lambda$, $\mu \in M$ and $\eta \in N$ consider the following quasiequilibrium problem:

$(\mathrm{P}_{\mathrm{r}\varphi})$ Find $\bar{x} \in K(\bar{x}, \lambda)$ and $\bar{x}^* \in a(\bar{x}, \eta)$ such that, for each $y \in K(\bar{x}, \lambda)$,

$$\mathrm{r}\big(F(\bar{x}, y, \bar{x}^*, \mu), \varphi(C)\big).$$

Let $S_{\mathrm{r}\varphi}(\lambda, \mu, \eta)$ be the solution set of $(\mathrm{P}_{\mathrm{r}\varphi})$ corresponding to λ, μ and η. Note that this problem statement is not quite explicit. However, it helps to unify the statements and proofs of assertions for four problems, $(\mathrm{P}_{\mathrm{r}\varphi})$ represents for each (λ, μ, η).

The following Hölder-related notions. are in use in the sequel.

Definition 1.1

(i) (Classical) A multifunction $G : X \times \Lambda \to 2^X$ is said to be $(l_1.\alpha_1, l_2.\ \alpha_2)-$Hölder at (x_0, λ_0) if there exist neighborhoods N of x_0 and U of λ_0 such that, $\forall x_1, x_2 \in N, \forall \lambda_1, \lambda_2 \in U$,

$$G(x_1, \lambda_1) \subseteq \left\{ x \in X \mid \exists z \in G(x_2, \lambda_2), d(x, z) \leq l_1 d^{\alpha_1}(x_1, x_2) + l_2 d^{\alpha_2}(\lambda_1, \lambda_2) \right\}.$$

(ii) (Anh and Khanh (2006)) $G : X \times X \to 2^Y$ is called $h.\beta-$Hölder - strongly pseudomonotone of the first type in $S \subseteq X$ if, $\forall x, y \in S : x \neq y$,

$$[G(x, y) \not\subseteq -\mathrm{int}C] \Longrightarrow [G(y, x) + hB\big(0, d^{\beta}(x, y)\big) \subseteq -C], \qquad (1)$$

where $h \geq 0$ and $\beta > 0$. G is called $h.\beta-$Hölder-strongly pseudomonotone of the second type if (1) is replaced by

$$[G(x, y) \subseteq Y \setminus -\mathrm{int}C] \Longrightarrow [G(y, x) + hB\big(0, d^{\beta}(x, y)\big) \subseteq -C].$$

(iii) (Bianchi and Schaible (1996)) Let $f : X \times X \to R$. f is called quasimonotone in $K \subseteq X$, if, $\forall x, y \in K : x \neq y$,

$$[f(x, y) < 0] \Longrightarrow [f(y, x) \geq 0].$$

f is called $h.\beta$- Hölder - strongly monotone in $K \subseteq X$ if, $\forall x, y \in K : x \neq y$,

$$f(x, y) + f(y, x) + hd^{\beta}(x, y) \leq 0.$$

4

The following Hölder-related assumptions (cf. Anh and Khanh (JOGO) online 06) will be essential for considering problem $(P_{r\varphi})$.

For the reference point $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$, there are neighborhoods $U(\lambda_0), V(\mu_0)$ and $W(\eta_0)$ of λ_0, μ_0 and η_0, respectively, such that

(A1) $\forall \lambda \in U(\lambda_0), \forall \mu_1, \mu_2 \in V(\mu_0), \forall x, y \in E(\lambda) := \{x \in A \mid x \in K(x, \lambda)\}$: $x \neq y, \forall x_1^*, x_2^* \in a\big(E(\lambda), W(\eta_0)\big),$

$$F(x, y, x_1^*, \mu_1) \subseteq F(x, y, x_2^*, \mu_2) + B\Big(0, d^\theta(x, y)\big(n_3 d^{\delta_3}(x_1^*, x_2^*) + n_4 d^{\delta_4}(\mu_1, \mu_2)\big)\Big),$$

where $n_3, n_4, \delta_3, \delta_4$ and θ are nonnegative real numbers.

$(A2_{r_1\varphi})$ $\forall \mu \in V(\mu_0), \forall \eta \in W(\eta_0), \forall x, y \in E\big(U(\lambda_0)\big) : x \neq y,$

$$hd^\beta(x, y) \leq \inf_{x^* \in a(x, \eta)} \inf_{g \in F(x, y, x^*, \mu)} d\big(g, \varphi(C)\big)$$

$$+ \inf_{y^* \in a(y, \eta)} \inf_{f \in F(y, x, y^*, \mu)} d\big(f, \varphi(C)\big), \qquad (2)$$

where $h > 0, \beta > \theta$.

$(A2_{r_2\varphi})$ is $(A2_{r_1\varphi})$ with (2) replaced by

$$hd^\beta(x, y) \leq \inf_{x^* \in A(x, \eta)} \sup_{g \in F(x, y, x^*, \mu)} d\big(g, \varphi(C)\big).$$

$$+ \inf_{y^* \in A(y, \eta)} \sup_{f \in F(y, x, y^*, \mu)} d\big(f, \varphi(C)\big).$$

Remark 1.1   These assumptions look seemingly complicated. But they are not hard to be checked as shown by examples below. We now make their meanings clearer.

(i)   Assumption (A1) incorporates Hölder continuity with respect to state variables x, y and to parameter μ (in connection also with parameters λ and η).

As explained in Anh and Khanh ((JOGO) online 2006), this condition replaces particular orthogonality and linearity of variational inequalities in Hilbert spaces to ensure the Hölder continuity of the solution (see Theorem 2.1).

(ii)   When $\varphi = \varphi_2$, assumptions (A2$_{\mathrm{r}\varphi_2}$) become assumptions (A2a) and (A2b) in Anh- Khanh ((JOGO) online 2006).

(iii)   To explain Assumption (A2$_{\mathrm{r}\varphi}$) we consider a single-valued real function (without parameters) $f : X \times X \to R$ for the sake of simplicity. Then the four assumptions (A2$_{\mathrm{r}\varphi}$) collapse to the following assumption: $\forall x, y \in K \subseteq X : x \neq y$,

$$hd^\beta(x, y) \leq d\big(f(x, y), R_+\big) + d\big(f(y, x), R_+\big). \tag{3}$$

We have the following relation.

Proposition 1.1 (Anh and Khanh ((JOGO) online 2006), Proposition 1.1).

(i)   *If $f : X \times X \to R$ satisfies (3) then f is $h.\beta-$Hölder-strongly pseudomonotone in K (the two types defined in Definition 1.1 (ii) coincide in this case). Conversely, if f is $h.\beta-$Hölder-strongly pseudomonotone in K and quasi-monotone in K, then f satisfies (3).*

(ii)   *If $f : X \times X \to R$ is $h.\beta-$Hölder-strongly monotone in $K \subseteq X$, then f satisfies (3).*

Examples 1.1 and 1.2 in Anh and Khanh ((JOGO) online 2006) interpret the lacking implications in Proposition 1.1.

In this paper we use mainly the following distance, for $C, D \subseteq X$,

$$\rho(C, D) = \sup_{x \in C, y \in D} d(x, y).$$

To compare the results we recall that the Hausdorff distance $H(C, D)$ and the

r-Hausdorff distance (see Attouch and Wets (1991)) $H_r(C,D)$ are defined as

$$H(C,D) = \max\{\sup_{x \in C} d(x,D), \sup_{y \in D} d(y,C)\},$$

$$H_r(C,D) = \max\{\sup_{x \in C \cap B(0,r)} d(x,D), \sup_{y \in D \cap B(0,r)} d(y,C)\}.$$

It is easy to see that the function $r \mapsto H_r$ is increasing and $\forall C, D \subseteq X$,

$$\rho(C,D) \geq H(C,D) = \lim_{r \to +\infty} H_r(C,D),$$

and, $\forall r > 0$,

$$\rho(C,D) \geq H(C,D) \geq H_r(C,D).$$

2   The main result

Theorem 2.1 *For problem* $(\mathrm{P_{r\varphi}})$ *assume that solutions exist in a neighborhood of the considered point* $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$ *and the assumptions* (A1) *and* $(\mathrm{A2_{r\varphi}})$ *are satisfied. Assume further that*

(i) *$K(.,.)$ is $(l_1.\alpha_1, l_2.\alpha_2) - $Hölder in $E(U(\lambda_0)) \times \lambda_0$;*

(ii) *$\forall (\lambda, \mu, \eta) \in U(\lambda_0) \times V(\mu_0) \times W(\eta_0)$, $\forall x \in E(\lambda)$, $\forall x^* \in a(x, \eta)$, $F(x, ., x^*, \mu)$ is $n_2.\delta_2 - $Hölder in $K\big(U(\lambda_0), \lambda\big)$;*

(iii) *$\forall x \in E(U(\lambda_0))$, $a(x,.)$ is $m.\gamma - $Hölder at η_0;*

(iv) *$\alpha_1 \delta_2 = \beta, h > 2n_2 l_1^{\delta_2}, \beta > \theta$.*

Then, the solution $x(\lambda, \mu, \eta)$ of $(\mathrm{P_{r\varphi}})$ is unique in a neighborhood of $(\lambda_0, \mu_0, \eta_0)$ and satisfies the following condition

$$d\big(x(\lambda_1, \mu_1, \eta_1), x(\lambda_2, \mu_2, \eta_2)\big) \leq k_1 d^{\alpha_2 \delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta_4/(\beta-\theta)}(\mu_1, \mu_2)$$

$$+ k_3 d^{\gamma \delta_3/(\beta-\theta)}(\eta_1, \eta_2),$$

7

where k_1, k_2 and k_3 are positive constants depending on $h, \beta, n_2, n_3, n_4, \theta, l_1, l_2, ...$

Proof   Since r $\in \{r_1, r_2\}$ and $\varphi \in \{\varphi_1, \varphi_2\}$, we have in fact four cases corresponding to four different combinations of values of r and φ. However, the proof techniques are similar. We consider only the case where r $= r_1$ and $\varphi = \varphi_2$. Let $\lambda_1, \lambda_2 \in U(\lambda_0), \mu_1, \mu_2 \in V(\mu_0)$ and $\eta_1, \eta_2 \in W(\eta_0)$.

Step 1   We prove that, $\forall x(\lambda_1, \mu_1, \eta_1) \in S_{r_1\varphi_2}(\lambda_1, \mu_1, \eta_1), \forall x(\lambda_1, \mu_2, \eta_1) \in S_{r_1\varphi_2}(\lambda_1, \mu_2, \eta_1)$,

$$d_1 := d\big(x(\lambda_1, \mu_1, \eta_1), x(\lambda_1, \mu_2, \eta_1)\big) \leq \left(\frac{n_4}{h - 2n_2 l_1^{\delta_2}}\right)^{1/(\beta-\theta)} d^{\delta_4/(\beta-\theta)}(\mu_1, \mu_2).$$

Let $x(\lambda_1, \mu_1, \eta_1) \neq x(\lambda_1, \mu_2, \eta_1)$ (if the equality holds then we are done). As $x(\lambda_1, \mu_1, \eta_1) \in K(x(\lambda_1, \mu_1, \eta_1), \lambda_1), x(\lambda_1, \mu_2, \eta_1) \in K(x(\lambda_1, \mu_2, \eta_1), \lambda_1)$ and $K(.,.)$ is Hölder continuous, there are $x_1 \in K(x(\lambda_1, \mu_1, \eta_1), \lambda_1)$ and $x_2 \in K(x(\lambda_1, \mu_2, \eta_1), \lambda_1)$ such that

$$d\big(x(\lambda_1, \mu_1, \eta_1), x_2\big) \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_1, \eta_1), x(\lambda_1, \mu_2, \eta_1)\big), \tag{4}$$

$$d\big(x(\lambda_1, \mu_2, \eta_1), x_1\big) \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_1, \eta_1), x(\lambda_1, \mu_2, \eta_1)\big). \tag{5}$$

Since $x(\lambda_1, \mu_1, \eta_1)$ and $x(\lambda_1, \mu_2, \eta_1)$ are solutions of $(\mathrm{P}_{r_1\varphi_2})$, there exist $x_1^* \in a(x(\lambda_1, \mu_1, \eta_1), \eta_1)$ and $x_2^* \in a(x(\lambda_1, \mu_2, \eta_1), \eta_1)$ such that

$$\exists z_1 \in F(x(\lambda_1, \mu_1, \eta_1), x_1, x_1^*, \mu_1) \cap (Y \setminus -\mathrm{int}C), \tag{6}$$

$$\exists z_2 \in F(x(\lambda_1, \mu_2, \eta_1), x_2, x_2^*, \mu_2) \cap (Y \setminus -\mathrm{int}C). \tag{7}$$

Assumption $(\mathrm{A2}_{r_1\varphi_2})$ implies that

$$\inf_{x^* \in a(x(\lambda_1,\mu_1,\eta_1),\eta_1)} \inf_{g \in F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x^*,\mu_1)} d(g, Y \setminus -\mathrm{int}C)$$

$$+ \inf_{x^* \in a(x(\lambda_1,\mu_2,\eta_1),\eta_1)} \inf_{f \in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x^*,\mu_1)} d(f, Y \setminus -\mathrm{int}C) \geq h d_1^\beta.$$

By (6) and (7), we have

$$\inf_{g\in F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_1^*,\mu_1)} d(g,z_1) + \inf_{f\in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_1)} d(f,z_2) \geq hd_1^\beta.$$

Hence,

$$H\big(F(x(\lambda_1,\mu_1,\eta_1),x_1,x_1^*,\mu_1),F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_1^*,\mu_1)\big)$$

$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x_2,x_2^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_1)\big) \geq hd_1^\beta,$$

where $H(.,.)$ is the Hausdorff distance. Consequently,

$$H\big(F(x(\lambda_1,\mu_1,\eta_1),x_1,x_1^*,\mu_1),F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_1^*,\mu_1)\big)$$

$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x_2,x_2^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_1)\big)$$

$$\geq hd_1^\beta.$$

By assumption (A1) and (ii), one has

$$n_2 d^{\delta_2}(x_1,x(\lambda_1,\mu_2,\eta_1)) + n_2 d^{\delta_2}(x_2,x(\lambda_1,\mu_1,\eta_1)) + n_4 d_1^\theta d^{\delta_4}(\mu_1,\mu_2) \geq hd_1^\beta.$$

Now (4) and (5) imply that

$$n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} + n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} + n_4 d_1^\theta d^{\delta_4}(\mu_1,\mu_2) \geq hd_1^\beta.$$

Then assumption (iv) yields that

$$d_1^{\beta-\theta} \leq \left(\frac{n_4}{h-2n_2 l_1^{\delta_2}}\right) d^{\delta_4}(\mu_1,\mu_2).$$

Setting $k_1 = \left(\frac{n_4}{h-2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta-\theta}}$, we have

$$d_1 \leq k_1 d^{\frac{\delta_4}{\beta-\theta}}(\mu_1,\mu_2).$$

Step 2 Now we show that, $\forall x(\lambda_1, \mu_2, \eta_1) \in S_{\mathrm{r}_1\varphi_2}(\lambda_1, \mu_2, \eta_1), \forall x(\lambda_2, \mu_2, \eta_1) \in S_{\mathrm{r}_1\varphi_2}(\lambda_2, \mu_2, \eta_1)$,

$$d_2 := d\big(x(\lambda_1, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_1)\big) \leq \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}}\right)^{1/\beta} d^{\alpha_2\delta_2/\beta}(\lambda_1, \lambda_2).$$

As before we can assume that $x(\lambda_1, \mu_2, \eta_1) \neq x(\lambda_2, \mu_2, \eta_1)$. Thanks to (i) we have $x_1' \in K(x(\lambda_2, \mu_2, \eta_1), \lambda_1)$ and $x_2' \in K(x(\lambda_1, \mu_2, \eta_1), \lambda_2)$ such that

$$d\big(x(\lambda_1, \mu_2, \eta_1), x_2'\big) \leq l_2 d^{\alpha_2}(\lambda_1, \lambda_2), \tag{8}$$

$$d\big(x(\lambda_2, \mu_2, \eta_1), x_1'\big) \leq l_2 d^{\alpha_2}(\lambda_1, \lambda_2). \tag{9}$$

By the Hölder continuity of $K(.,.)$ there are $x_1'' \in K(x(\lambda_1, \mu_2, \eta_1), \lambda_1)$ and $x_2'' \in K(x(\lambda_2, \mu_2, \eta_1), \lambda_2)$,

$$d(x_1', x_1'') \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_1)\big), \tag{10}$$

$$d(x_2', x_2'') \leq l_1 d^{\alpha_1}\big(x(\lambda_1, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_1)\big). \tag{11}$$

By the definition of $(\mathrm{P}_{\mathrm{r}_1\varphi_2})$, $x_1''^* \in a(x(\lambda_1, \mu_2, \eta_1), \eta_1)$ and $x_2''^* \in a(x(\lambda_2, \mu_2, \eta_1), \eta_1)$ exist such that one can find

$$z_1' \in F(x(\lambda_1, \mu_2, \eta_1), x_1'', x_1''^*, \mu_2) \cap (Y \setminus -\mathrm{int}C), \tag{12}$$

$$z_2' \in F(x(\lambda_2, \mu_2, \eta_1), x_2'', x_2''^*, \mu_2) \cap (Y \setminus -\mathrm{int}C). \tag{13}$$

It follows from assumption $(\mathrm{A2}_{\mathrm{r}_1\varphi_2})$ that

$$\inf_{x^* \in a(x(\lambda_1, \mu_2, \eta_1), \eta_1)} \inf_{g \in F(x(\lambda_1, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_1), x^*, \mu_2)} d(g, Y \setminus -\mathrm{int}C)$$

$$+ \inf_{x^* \in a(x(\lambda_2, \mu_2, \eta_1), \eta_1)} \inf_{f \in F(x(\lambda_2, \mu_2, \eta_1), x(\lambda_1, \mu_2, \eta_1), x^*, \mu_2)} d(f, Y \setminus -\mathrm{int}C) \geq h d_2^\beta.$$

(12) and (13) then imply that

$$\inf_{g \in F(x(\lambda_1, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_1), x_1''^*, \mu_2)} d(g, z_1') + \inf_{f \in F(x(\lambda_2, \mu_2, \eta_1), x(\lambda_1, \mu_2, \eta_1), x_2''^*, \mu_2)} d(f, z_2') \geq h d_2^\beta.$$

Consequenly,

$$H\big(F(x(\lambda_1,\mu_2,\eta_1),x_1'',x_1''^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1),x_1''^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_2,\mu_2,\eta_1),x_2'',x_2''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_2''^*,\mu_2)\big) \geq hd_2^\beta.$$

and hence

$$H\big(F(x(\lambda_1,\mu_2,\eta_1),x_1'',x_1''^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x_1',x_1''^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x_1',x_1''^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1),x_1''^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_2,\mu_2,\eta_1),x_2'',x_2''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_1),x_2',x_2''^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_2,\mu_2,\eta_1),x_2',x_2''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_2''^*,\mu_2)\big)$$

$$\geq hd_2^\beta.$$

The Hölder continuity of F assumed in (ii) implies that

$$n_2 d^{\delta_2}(x_1'',x_1') + n_2 d^{\delta_2}(x_1',x(\lambda_2,\mu_2,\eta_1)) + n_2 d^{\delta_2}(x_2'',x_2')$$

$$+n_2 d^{\delta_2}(x_2',x(\lambda_1,\mu_2,\eta_1)) \geq hd_2^\beta.$$

From (8), (9), (10) and (11) we have

$$n_2 l_1^{\delta_2} d_2^{\alpha_1\delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2\delta_2}(\lambda_1,\lambda_2) + n_2 l_1^{\delta_2} d_2^{\alpha_1\delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2\delta_2}(\lambda_1,\lambda_2) \geq hd_2^\beta.$$

It follows from assumption (iv) that

$$d_2^\beta \leq \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}}\right) d^{\alpha_2\delta_2}(\lambda_1,\lambda_2).$$

Taking $k_2 = \left(\frac{2n_2 l_2^{\delta_2}}{h-2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta}}$, one has

$$d_2 \leq k_2 d^{\frac{\alpha_2\delta_2}{\beta}}(\lambda_1,\lambda_2).$$

Step 3 We check the inequality, $\forall x(\lambda_2,\mu_2,\eta_1) \in S_{r_1\varphi_2}(\lambda_2,\mu_2,\eta_1)$, $\forall x(\lambda_2,\mu_2,\eta_2) \in S_{r_1\varphi_2}(\lambda_2,\mu_2,\eta_2)$,

$$d_3 := d\big(x(\lambda_2,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_2)\big) \leq \left(\frac{n_3 m^{\delta_3}}{h - 2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma\delta_3}{\beta-\theta}}(\eta_1,\eta_2).$$

11

Assume that $x(\lambda_2, \mu_2, \eta_1) \neq x(\lambda_2, \mu_2, \eta_2)$. It follows from (i) the existence of $x_1''' \in K(x(\lambda_2, \mu_2, \eta_1), \lambda_2)$ and $x_2''' \in K(x(\lambda_2, \mu_2, \eta_2), \lambda_2)$ such that

$$d\big(x(\lambda_2, \mu_2, \eta_1), x_2'''\big) \leq l_1 d_3^{\alpha_1}, \tag{14}$$

$$d\big(x(\lambda_2, \mu_2, \eta_2), x_1'''\big) \leq l_1 d_3^{\alpha_1}. \tag{15}$$

Since $x(\lambda_1, \mu_1, \eta_1)$ and $x(\lambda_1, \mu_2, \eta_1)$ are solutions of $(\mathrm{P}_{\mathrm{r}_1\varphi_2})$, there exist $x_1'''^* \in a(x(\lambda_2, \mu_2, \eta_1), \eta_1)$ and $x_2'''^* \in a(x(\lambda_2, \mu_2, \eta_2), \eta_2)$ such that we have

$$z_1'' \in F(x(\lambda_2, \mu_2, \eta_1), x_1''', x_1'''^*, \mu_2) \cap (Y \setminus -\mathrm{int}C), \tag{16}$$

$$z_2'' \in F(x(\lambda_2, \mu_2, \eta_2), x_2''', x_2'''^*, \mu_2) \cap (Y \setminus -\mathrm{int}C). \tag{17}$$

Assumption $(\mathrm{A2}_{\mathrm{r}_1\varphi_2})$ implies that

$$\inf_{x^* \in a(x(\lambda_2,\mu_2,\eta_1),\eta_1)} \inf_{g \in F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_2),x^*,\mu_2)} d(g, Y \setminus -\mathrm{int}C)$$

$$+ \inf_{x^* \in a(x(\lambda_2,\mu_2,\eta_2),\eta_1)} \inf_{f \in F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x^*,\mu_2)} d(f, Y \setminus -\mathrm{int}C) \geq h d_3^{\beta}. \tag{18}$$

Since $x_2''' \in a(x(\lambda_2, \mu_2, \eta_2), \eta_2)$, by (iii) there exists $x_1^* \in a(x(\lambda_2, \mu_2, \eta_2), \eta_1)$ such that

$$d(x_2''', x_1^*) \leq m d^{\gamma}(\eta_1, \eta_2). \tag{19}$$

It follows from (18) that

$$\inf_{g \in F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_2),x_1'''^*,\mu_2)} d(g, Y \setminus -\mathrm{int}C)$$

$$+ \inf_{f \in F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x_1^*,\mu_2)} d(f, Y \setminus -\mathrm{int}C) \geq h d_3^{\beta}.$$

From (16) and (17) one has

$$\inf_{g \in F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_2),x_1'''^*,\mu_2)} d(g, z_1'') + \inf_{f \in F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x_1^*,\mu_2)} d(f, z_2'') \geq h d_3^{\beta}.$$

Hence

$$H\big(F(x(\lambda_2, \mu_2, \eta_1), x_1''', x_1'''^*, \mu_2), F(x(\lambda_2, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_2), x_1'''^*, \mu_2)\big)$$

$$+H\big(F(x(\lambda_2,\mu_2,\eta_2),x_2''',x_2'''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x_1^*,\mu_2)\big) \geq hd_3^\beta,$$

and then

$$H\big(F(x(\lambda_2,\mu_2,\eta_1),x_1''',x_1'''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_2),x_1'''^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_2,\mu_2,\eta_2),x_2''',x_2'''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x_2'''^*,\mu_2)\big)$$

$$+H\big(F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x_2'''^*,\mu_2),F(x(\lambda_2,\mu_2,\eta_2),x(\lambda_2,\mu_2,\eta_1),x_1^*,\mu_2)\big)$$

$$\geq hd_3^\beta.$$

Assumptions (A1) and (ii) together imply that

$$n_2 d^{\delta_2}(x_1''',x(\lambda_2,\mu_2,\eta_2)) + n_2 d^{\delta_2}(x_2''',x(\lambda_2,\mu_2,\eta_1)) + n_3 d_3^\theta d^{\delta_3}(x_2'''^*,x_1^*) \geq hd_3^\beta.$$

From (14), (15) and (19), we obtain

$$n_2 l_1^{\delta_2} d_3^{\alpha_1\delta_2} + n_2 l_1^{\delta_2} d_3^{\alpha_1\delta_2} + n_3 m^{\delta_3} d_3^\theta d^{\gamma\delta_3}(\eta_1,\eta_2) \geq hd_3^\beta.$$

Assumption (iv) now yields that

$$d_3^{\beta-\theta} \leq \left(\frac{n_3 m^{\delta_3}}{h - 2n_2 l_1^{\delta_2}}\right) d^{\gamma\delta_3}(\eta_1,\eta_2).$$

Setting $k_3 = \left(\frac{n_3 m^{\delta_3}}{h-2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta-\theta}}$, we have

$$d_3 \leq k_3 d^{\frac{\gamma\delta_3}{\beta-\theta}}(\mu_1,\mu_2).$$

Step 4 Finally since, $\forall x(\lambda_1,\mu_1,\eta_1) \in S_{r_1\varphi_2}(\lambda_1,\mu_1,\eta_1)$, $\forall x(\lambda_2,\mu_2,\eta_2) \in S_{r_1\varphi_2}(\lambda_2,\mu_2,\eta_2)$

$$d\big(x(\lambda_1,\mu_1,\eta_1),x(\lambda_2,\mu_2,\eta_2)\big) \leq d_1 + d_2 + d_3,$$

we have

$$\rho\big(S_{r_1\varphi_2}(\lambda_1,\mu_1,\eta_1),S_{r_1\varphi_2}(\lambda_2,\mu_2,\eta_2)\big) \leq d_1 + d_2 + d_3.$$

Putting $(\lambda_1,\mu_1,\eta_1) = (\lambda_2,\mu_2,\eta_2)$ from this inequality one sees that $S_{r_1\varphi_2}(\lambda_1,\mu_1,\eta_1)$ is a singleton. Similarly, $S_{r_1\varphi_2}(\lambda_2,\mu_2,\eta_2)$ is also a singleton. Thus $(P_{r_1\varphi_2})$ has a unique solution in a neighborhood of (λ_0,μ_0,η_0) and then the Hölder condition concluded in the theorem is obtained. $\qquad\square$

Remark 2.1 In the case of equilibrium problems considered in Anh and Khanh ((JOGO) online 2006), generalized monotonicity assumptions corresponding to ($A2_{r\varphi}$) ensure directly the solution uniqueness. However, for quasiequilibrium problems the above proof shows that this uniqueness is obtained by invoking all the assumptions together.

Examples 2.1 in Anh and Khanh ((JOGO) online) shows that assumptions ($A2_{r\varphi}$) are essential even in the special case where K depends only on λ.

Now we discuss some consequences of Theorem 2.1 for this special case, i.e. problems ($P_{r\varphi}$) becomes the corresponding equilibrium problem denoted by ($E_{r\varphi}$). When r = $r_1, \varphi = \varphi_2$ and r = $r_2, \varphi = \varphi_2$, Theorem 2.1 becomes Theorems 2.1 and 2.2, respectively, of Anh and Khanh ((JOGO) online 2006) and sharpens Theorems 2.1 and 2.2, respectively, of Anh and Khanh (2006). To see this sharpening see Examples 2.4-2.6 in Anh and Khanh ((JOGO) online 2006). Note that the cases where r = $r_1, \varphi = \varphi_1$ or r = $r_2, \varphi = \varphi_1$ are new even for the special case of ($E_{r\varphi}$). In addition, if $a(x, \eta) \equiv \{x\}$ and $F(x, y, x^*, \mu) = F(x, y, \mu)$ is single-valued, the special case of Theorem 2.1 improves Theorem 4.2 of Bianchi and Pini (2003) and Theorem 2.2.1 of Ait Mansour and Riahi (2005) (see also Examples 2.4 - 2.8 of Anh and Khanh ((JOGO) online 2006) for detailed comparisons).

3   Applications

We will now apply the main result in Section 2 to some problems of importance. Since quasiequilibrium problems include also many other problems, our result can clearly imply consequences for them.

3.1 Multivalued quasivariational inequalities

In this subsection, if not stated otherwise, let X be a reflexive Banach space, N and Λ be metric linear spaces, and $A \subseteq X$ be a nonempty subset. Let $K : A \times \Lambda \to 2^X$ and $a : A \times N \to 2^{X^*}$ be multifunctions with $K(x, \lambda)$ being closed and convex, $\forall (x, \lambda) \in A \times \Lambda$. For each $(\lambda, \eta) \in \Lambda \times N$ consider the quasivariational inequality problem

(QVI) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that $\exists \bar{t} \in a(\bar{x}, \eta)$, $\forall y \in K(\bar{x}, \lambda)$,

$$\langle \bar{t}, y - \bar{x} \rangle \geq 0.$$

For each $(\lambda, \eta) \in \Lambda \times N$, by $S_{\mathrm{vi}}(\lambda, \eta)$ we denote the solution set of (QVI) at (λ, η).

To convert (QVI) to a special case of $(\mathrm{P}_{\mathrm{r}\varphi})$ set $Z = X^*, Y = R, C = R_+$ and $F(x, y, x^*) = \langle x^*, y - x \rangle$.

Corollary 3.1 *For* (QVI) *assume the solution existence in a neighborhood of* $(\lambda_0, \eta_0) \in \Lambda \times N$. *Assume further that there are neighborhoods* $U(\lambda_0)$ *of* λ_0, $W(\eta_0)$ *of* η_0 *such that we have* (i), (iii) *of Theorem 2.1 and*

(A2) $\forall \eta \in W(\eta_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$h\|x - y\|^\beta \leq \inf_{g \in \langle a(x,\eta), y-x \rangle} d(g, R_+) + \inf_{f \in \langle a(y,\eta), x-y \rangle} d(f, R_+);$$

  (a) *a is bounded in* $E(U(\lambda_0)) \times \{\eta_0\}$: $\|a(x, \eta)\| \leq n_2, \forall x \in E(U(\lambda_0)), \forall \eta \in W(\eta_0)$ *and* $E(U(\lambda_0))$ *is bounded;*

  (b) $\alpha_1 = \beta, h > 2n_2 l_1$.

Then the solution $x(\lambda, \eta)$ *of* (QVI) *is unique in a neighborhood of* (λ_0, η_0) *and*

satisfies the Hölder condition

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \leq k_1 d^{\alpha_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\gamma/\beta}(\mu_1, \mu_2).$$

Proof We simply check the assumptions of Theorem 2.1, except (i) and (iii). $(A2_{r\varphi})$ collapses to (A2) in this special case. For (A1) we see that

$$|F(x, y, x_1^*) - F(x, y, x_2^*)| = |\langle x_1^*, y - x \rangle - \langle x_2^*, y - x \rangle|$$

$$\leq \|y - x\| \|x_1^* - x_2^*\| \leq n_1 \|x_1^* - x_2^*\|.$$

Hence (A1) is fulfilled with $n_3 = n_1, \delta_3 = 1$ and $\theta = n_4 = \delta_4 = 0$. Since $\|a(x, \eta)\| \leq n_2$ in $E(U(\lambda_0)) \times W(\eta_0)$, assumption (ii) is satisfied with n_2 and $\delta_2 = 1$. Assumption (iv) becomes (b) in this case. □

Remark 3.1 Let $\bar{x} = x(\bar{\lambda}, \bar{\mu})$ be the solution of the variational inequality (VI) corresponding to (QVI), i.e. when K does not depend on x. Using similar arguments, Corollary 3.1 can be proved when replacing assumption (i) by the following Aubin property (known also as pseudo-Lipschitz property) of K around $(\bar{x}, \bar{\lambda})$ (but we have to add the maximal monotonicity of $a(., \eta)$): there exist neighborhoods P of $\bar{x}$, $\mathcal{V}(\bar{\lambda})$ of $\bar{\lambda}$ and $k > 0$ such that, $\forall \lambda_1, \lambda_2 \in \mathcal{V}(\bar{\lambda})$,

$$K(\lambda_1) \cap P \subseteq K(\lambda_2) + lB(0, d(\lambda_1, \lambda_2)).$$

Indeed, by classical arguments of existence theory for variational inequalities (cf. Anh and Khanh (JOGO) online 2006), we can consider that the solution $x(\lambda, \eta)$ to (VI) belongs to $K(\bar{\lambda}) \cap P$. Note that in this case $l_1 = 0$ and α_1 is arbitrary so we take $\alpha_1 = \beta$ and hence $\alpha_1.\delta_2 = \beta > 1$. Furthermore, in this case assumption (a) of Corollary 3.1 requires only a to be bounded in $E(U(\lambda_0)) \times \{\eta_0\}$ $(E(U(\lambda_0))$ needs not to be bounded) and hence assumption (A1) will be satisfied with $n_3 =$

$\delta_3 = \theta = 1$ and $n_4 = \delta_4 = 0$. Namely we have the following consequence.

Corollary 3.2 *For* (VI) *assume the existence a neighborhood* $U(\lambda_0) \times W(\eta_0)$ *of* $(\lambda_0, \eta_0) \in \Lambda \times N$ *such that assumptions* (iii) *and* (A2) *of Corollary 3.1 are satisfied and assume further that*

(i') *there is a neighborhood* P *of the solution* $x(\lambda_0, \eta_0)$ *such that,* $\forall \lambda, \lambda' \in U(\lambda_0)$,

$$K(\lambda) \cap P \subseteq K(\lambda') + lB\big(0, d^\alpha(\lambda, \lambda')\big)$$

    (*i.e.* $K(.)$ *is* $l.\alpha-pseudo$-*Hölder at* λ_0);

(a') a *is bounded in* $K\big(U(\lambda_0)\big) \times \{\eta_0\}$ *and* $\forall \eta \in W(\eta_0)$, $a(., \eta)$ *is maximal monotone;*

(b') $\beta > 1$.

Then, in a neighborhood of (λ_0, η_0), *the solution* $x(\lambda, \eta)$ *of* (VI) *is unique and satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \leq k_1 d^{\alpha/\beta}(\lambda_1, \lambda_2) + k_2 d^{\xi/(\beta-1)}(\eta_1, \eta_2).$$

In the case where a is single - valued, Corollary 3.2 implies the following result

Corollary 3.3 *For* (VI) *assume that* a *is single - valued,* $x_0 := x(\lambda_0, \eta_0)$ *is a solution of* (VI) *at* (λ_0, η_0) *and that there is a neighborhood* $U(\lambda_0) \times W(\eta_0)$ *of* (λ_0, η_0) *such that*

(A2') $a(., \eta)$ *is strongly monotone for each* $\eta \in W(\eta_0)$;

(i') $K(.)$ *is pseudo-Lipschitz in* $U(\lambda_0)$;

(iii') $a(., .)$ *is Lipschitz in* $P(x_0) \times W(\eta_0)$.

Then, in a neighborhood of (λ_0, η_0), *the unique solution of* (VI) *satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \le k_1 d^{1/2}(\lambda_1, \lambda_2) + k_2 d(\eta_1, \eta_2).$$

Proof　We check the assumptions of Corollary 3.1. (i') holds with $\alpha = 1$. (A2) is satisfied with $\beta = 2$ by (A2'). (iii) is fulfilled with $\gamma = 1$ by (iii'). For (a') we see that a is bounded since $a(.,.)$ is Lipschitz continuous and $a(.,\eta)$ is monotone; furthermore, since $a(.,.)$ is single-valued and $a(.,\eta)$ is monotone and demicontinuous, $a(.,\eta)$ is maximal monotone by Lemma 2.13 of Kluges (1979). Finally, assumption (b') is clearly satisfied.　□

If X is a Hilbert space Corollary 3.3 collapses to Theorem 2.1 of Yen (1995).

Remark 3.2　As shown by Proposition 1.1, if a is single-valued, assumption (A2) is more relaxed than the $h.\beta$-Hölder-strong monotonicity of $a(.,\eta)$ in $E(U(\lambda_0))$, $\forall \eta \in W(\lambda_0)$. When a is single-valued and $K(x, \lambda)$ is of a special linear form defined by the travel demands in a traffic network problem (see Subsection 3.2), our problem (QVI) is reduced to the problem investigated in Adly et al. (in press). Theorem 5.1, the main result there, under assumptions similar to that of Corollary 3.1 with the mentioned monotonicity of $a(.,\eta)$ replacing (A2), is weaker than Corollary 3.1 when $\beta = 2$ since the distance $H_r(C, D)$ is used instead of $\rho(C, D)$, and hence the solution uniqueness is not established.

3.2　A vector quasioptimization problem

Let X, Y, Λ, N, C and K be as for problem $(P_{r\varphi})$ in Section 1 and $D : X \times N \to 2^Y$ be a multifunction. For each $(\lambda, \eta) \in \Lambda \times N$, consider the following problem of

(VOP$_\varphi$)   finding $\bar{x} \in K(\bar{x}, \lambda)$ and $\bar{x}^* \in D(\bar{x}, \eta)$ such that, $\forall y \in K(\bar{x}, \lambda)$,

$$D(y, \eta) - \bar{x}^* \subseteq \varphi(C).$$

Recall that when K does not depend on x and $\varphi = \varphi_2$ such a point $\bar{x}$ is said to be a weak minimizer and $\bar{x}^*$ is a weak minimum of the vector optimization problem

$$\min D(y, \eta), s.t. \quad y \in K(\lambda),$$

and when K does not depend on x and $\varphi = \varphi_1$ such a point $\bar{x}$ is called an efficient minimizer and $\bar{x}^*$ is a Pareto minimum, i.e., there is no $y \in D(\bar{x}, \eta)$ such that

$$y - x^* \in (-C) \setminus l(C),$$

where $l(C) = C \cap (-C)$. Since the constraint set K here depends also on x, we have a quasioptimization problem.

To convert (VOP$_\varphi$) to a special case of (P$_{r_2\varphi}$) we simply set $Z = Y, M \equiv N$ and $F(x, y, x^*, \eta) = D(y, \eta) - x^*$. Then, from Theorem 2.1 we have (cf. also the proof of Theorem 2.1).

Corollary 3.4   *For* (VOP$_\varphi$) *assume that solutions exist in a neighborhood of* $(\lambda_0, \eta_0) \in \Lambda \times N$. *Assume further that there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $W(\eta_0)$ *of* η_0 *such that*

(A1) $\forall \lambda \in U(\lambda_0), \forall \eta_1, \eta_2 \in W(\eta_0), \forall y \in E(\lambda), \forall x_1^*, x_2^* \in D\big(E(\lambda), W(\eta_0)\big)$,

$$D(y, \eta_1) - x_1^* \subseteq D(y, \eta_2) - x_2^* + \|y\|^\theta B\big(0, n_3\|x_1^* - x_2^*\| + n_4 d^{\delta_4}(\eta_1, \eta_2)\big),$$

  where $n_3, n_4 > 0, \theta \geq 0$ *and* $\delta_4 > 0$;

(A2$_\varphi$) $\forall \eta \in W(\eta_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$hd^\beta(x, y) \leq \inf_{x^* \in D(x, \eta)} \sup_{g \in D(y, \eta) - x^*} d\big(g, \varphi(C)\big) + \inf_{y^* \in D(y, \eta)} \sup_{f \in D(x, \eta) - y^*} d\big(f, \varphi(C)\big),$$

  for $h > 0$ *and* $\beta > \theta$;

19

(i) $K(.,.)$ is $(l_1.\alpha_1, l_2.\alpha_2)-H\ddot{o}lder$ at λ_0;

(ii) $\forall \eta \in W(\eta_0)$, $D(.,\eta)$ is $n_2.\delta_2-$ $H\ddot{o}lder$ in $E(U(\lambda_0))$;

(iii) $\forall \lambda \in U(\lambda_0), \forall y \in E(\lambda), D(y,.)$ is $m.\gamma$ $H\ddot{o}lder$ at η_0.

Then, in a neighborhood of (λ_0, η_0), the solution $x(\lambda, \eta)$ of (VOP_φ), is unique and satisfies the following condition

$$d\big(x(\lambda_1, \eta_1), x(\lambda_2, \eta_2)\big) \leq k_1 d^{\alpha_2 \delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\tau/(\beta-\theta)}(\eta_1, \eta_2),$$

where $\tau := \min\{\delta_4, \gamma\}$, k_1 and k_2 are positive constants depending on $h, \beta, m, \theta, ...$

3.3 Applications to traffic network problems

Wardrop (1952) introduced a notion of equilibrium flows for transportation network problems and proved basic traffic network principles. Until now many contributions have developed this research direction in various aspects. We would notice some points in the development process. Smith (1979) began the variational approach by proving that the Wardrop equilibria are just solutions of variational inequalities corresponding to the traffic network problems. In De Luca (1995) and Maugeri (1995) the travel demands of the problem was proposed to depend on the equilibrium vector flows to meet the variety of practical situations. These elastic demands led to the fact that the traffic problem corresponded to a quasivariational (not variational) inequality. De Luca (1995) and Maugeri (1995) considered the travel costs being multifunctions of the path flows. Khanh and Luu (2004, 2005) extended the notion of Wardrop's traffic equilibria to this case. Efforts have been devoted mainly to the solution existence. Resently, in Adly et al. (in press), the stability of the problem in terms of the Hölder continuity of the solution sets with respect to the perturbing parameters is studied. Unfortunately,

some assumptions are mistakenedly imposed and contradict each other and so no problems satisfy them. In this subsection we apply the results in Section 2 to establish sharpened Hölder continuity results for more general traffic network problems with multivalued costs. We show also that the solution is locally unique.

We describe first our traffic problem. Let the network consist of nodes and links (or arcs). Let $W = (W_1, ..., W_l)$ be the set of pairs, each of them consists of an origin node and a destination node, (O/D pairs for short). Assume that P_j, $j = 1, ..., l$, is the set of paths connecting the pair W_j and that P_j contains $r_j \geq 1$ paths. Let $m = r_1 + ... + r_l$ and $f = (f_1, ..., f_m)$ denote the path vector flow. Giannessi proposed in Giannessi (1980) that restrictions of the capacity of the paths must be considered. Hence we assume that the constraint of the capacity of paths is of the form

$$A = \{f \in R^m : \gamma_s \leq f_s \leq \Gamma_s, s = 1, ..., m\},$$

where γ_s and Γ_s are given nonnegative numbers. Let the cost vector $T(f, \mu) = (T_1(f, \mu), ..., T_m(f, \mu))$ be a multifunction of flow f and perturbing parameter μ. The generalization of the Wardrop equilibrium for the multivalued cost case is as follows.

Definition 3.1 (i)   A path vector flow f is said to be a weak equilibrium vector flow if $\forall W_j, \forall q \in P_j, \forall s \in P_j, \exists t \in T(f, \mu)$,

$$t_q < t_s \Longrightarrow f_q = \Gamma_q \quad \text{or} \quad f_s = \gamma_s,$$

where $j = 1, ..., l$ and $q, s \in \{1, ..., m\}$ are among r_j paths corresponding to P_j.

(ii) A path vector flow f is called a strong equilibrium vector flow if (i) is satisfied with $\exists t \in T(f, \mu)$ being replaced by $\forall t \in T(f, \mu)$.

Assume further that the travel demand g_j of the O/D pair W_j depends on the equilibrium vector flow h as explained in De Luca (1995) and Maugeri (1995) and also on a perturbing parameter $\lambda \in \Lambda : g_j(h, \lambda)$. Denote the travel vector demand by $g = (g_1, ..., g_l)$ and use the Kronecker numbers

$$\phi_{js} = \begin{cases} 1, & \text{if } s \in P_j, \\ 0, & \text{if } s \notin P_j, \end{cases}$$

$$\phi = \{\phi_{js}\}, j = 1, ..., l; s = 1, ..., m.$$

Then the set of all feasible path vector flows is

$$K(h, \lambda) = \{f \in A \mid \phi z = g(h, \lambda)\},$$

where ϕ is called the O/D pair - path incidence matrix.

Note that the traffic problem formulated in terms of path flow variables as above needs not the additivity of the travel cost, i.e. a path cost may not be equal to the sum of the link costs for all links involved in the path. For formulations based on link variables such additivity must be assumed.

Observe that a feasible path flow vector $\bar{f}$ is a weak (or strong) equilibrium flow vector if $\bar{f}$ is a solution of the following quasivariational inequality, respectively, (see Khanh and Luu 2004):

(TNP_{r_1}) Find $\bar{f} \in K(\bar{f}, \lambda)$ such that, $\forall f \in K(\bar{f}, \lambda), \exists \bar{t} \in T(\bar{f}, \mu)$,

$$\langle \bar{t}, f - \bar{f} \rangle \geq 0.$$

(TNP_{r_2}) Find $\bar{f} \in K(\bar{f}, \lambda)$ such that $\forall f \in K(\bar{f}, \lambda), \forall \bar{t} \in T(\bar{f}, \mu)$ such that

$$\langle \bar{t}, f - \bar{f} \rangle \geq 0.$$

Lemma 3.5 (Proposition 5.1, Adly et al. (in press)) *If $g(., .)$ is $(L_1.\alpha_1, L_2.\alpha_2)$-Hölder at (x_0, λ_0) then there are l_1, l_2 such that K is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder at*

(x_0, λ_0).

Setting $X = Z = R_m, N \equiv \Lambda, Y = R, C = R_+, a(x, \lambda) = Z$ and $F(h, f, h^*, \mu) = \langle T(h, \mu), f - h \rangle$. Then our problems $(\mathrm{P}_{\mathrm{r}\varphi_1})$ coincides with $(\mathrm{P}_{\mathrm{r}\alpha_2})$; $(\mathrm{P}_{\mathrm{r}_1\varphi})$ and $(\mathrm{P}_{\mathrm{r}_2\varphi})$ becomes $(\mathrm{TNP}_{\mathrm{r}_1})$ and $(\mathrm{TNP}_{\mathrm{r}_2})$, respectively. Hence we can derive the Hölder continuity of $(\mathrm{TNP}_{\mathrm{r}_1})$ and $(\mathrm{TNP}_{\mathrm{r}_2})$ from Theorem 2.1 as follows.

First note that, since in this case $F(h, f, h^*, \mu)$ does not depend on h^* and the problem is scalar (hence $\varphi_1 = \varphi_2$), assumptions $(\mathrm{A}_{\mathrm{r}_1\varphi})$ and $(\mathrm{A}_{\mathrm{r}_2\varphi})$ collapse to the following assumptions, respectively,

$(\mathrm{A}_{\mathrm{r}_1})$ $\forall \mu \in V(\mu_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$h \|x - y\|^\beta \leq \inf_{g \in \langle T(x, \mu), y - x \rangle} d(g, R_+) + \inf_{f \in \langle T(y, \mu), x - y \rangle} d(f, R_+).$$

$(\mathrm{A}_{\mathrm{r}_2})$ $\forall \mu \in V(\mu_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$h \|x - y\|^\beta \leq \sup_{g \in \langle T(x, \mu), y - x \rangle} d(g, R_+) + \sup_{f \in \langle T(y, \mu), x - y \rangle} d(f, R_+).$$

Corollary 3.6 *For $(\mathrm{TNP}_{\mathrm{r}})$ assume that there are neighborhoods $U(\lambda_0)$ of λ_0 and $V(\mu_0)$ of μ_0 such that assumption $(\mathrm{A}_{\mathrm{r}})$ is satisfied. Assume further that*

(a) *$\forall f \in E(U(\lambda_0))$, $T(f, .)$ is $n.\delta$-Hölder at μ_0 and $T(., .)$ is bounded: $\forall f \in E(U(\lambda_0))$, $\forall \mu \in V(\mu_0)$, $\forall t \in T(f, \mu), \|t\| \leq M$; and $E(U(\lambda_0))$ is bounded: $\forall f \in E(U(\lambda_0))$, $\|f\| \leq N$;*

(b) *g is $(L_1.\alpha_1, L_2.\alpha_2)$-Hölder in $E(U(\lambda_0)) \times \{\lambda_0\}$;*

(c) *$\alpha_1 = \beta$ and $h > 2n_2 l_1$.*

Then, in a neighborhood of (λ_0, μ_0), the solution of $(\mathrm{TNP}_{\mathrm{r}})$ is unique and satisfies the following Hölder condition

$$d\big(f(\lambda_1, \mu_1), f(\lambda_2, \mu_2)\big) \leq k_1 d^{\alpha_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta/\beta}(\mu_1, \mu_2),$$

23

where k_1 and k_2 are positive constants depending on h, β, n, δ, etc.

Proof Taking into account Lemma 3.5, it suffices to check only assumptions (ii) and (A1) of Theorem 2.1 (assumption (iii) is satisfied with any $m \geq 0, \gamma \geq 0$). For any $F_1 = \langle t, f_1 - h \rangle$ we take $F_2 = \langle t, f_2 - h \rangle$. Then

$$\|F_1 - F_2\| \leq \langle t, f_1 - f_2 \rangle \leq M \|f_1 - f_2\|.$$

Hence

$$\langle T(h, \mu), f_1 - h \rangle \subseteq \langle T(h, \mu), f_2 - h \rangle + M \|f_1 - f_2\| B(0, 1),$$

i.e. assumption (ii) is fulfilled with $n_2 = M$ and $\delta_2 = 1$. Similarly, it is not hard to see that (A1) is satisfied with $\theta = 0, n_4 = nN, \delta_4 = \delta, n_3 = 0$ and δ_3 is any nonnegative numbers. □

Remark 3.3 In Adly et al. (in press) the special case of our traffic network problem, where T is single - valued, is investigated. Instead of (A_r) a strong monotonicity of T (which is stricter than (A_r)) is assumed. The other assumptions are similar to ours, but imposed globally, not only in $E(U(\lambda_0))$ as in Corollary 3.6. Furthermore, replacing our assumption (c), another technical assumption is used in Adly et al. (in press), using the lower bound $f_0 > 0$ of the norm $\|v\|$ of all $v \in A \setminus \{0\}$ (see Remark 3.1, Adly et al., in press), where A is assumed to be convex and contain 0. This assumption is mistaken, since there is no such convex set. Moreover, Adlly et al. (in press) uses the distance H_r instead of our distance ρ. Hence, the obtained Hölder condition for the solution set is weaker than the conclusion of Corollary 3.6 and cannot imply the solution uniqueness.

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